Lecture 5: Growth Theory

See Barro Ch. 3

Trevor Gallen

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Production Function-Intro

- Q: How do we summarize the production of five million firms all taking in different capital and labor types and producing different goods?
- A: “The” production function
- Assume a function that takes in capital and labor and spits out produced goods
- This will, and won't, be as offensive as it sounds

Production Function-I

- We typically write the production function:

$$Y \quad = \quad A \quad \cdot F(\quad K \quad , \quad L \quad)$$

- Y is production, GDP
- A is total factor productivity, productivity, or technology
- K is capital: all capital
- L is all labor hours
- Moreover, we frequently assume it to be "Cobb-Douglas"

$$Y = AK^{\alpha}L^{\beta}$$

- Where α and β are the contributions to GDP of capital and labor, respectively
- We'll see this will be equivalent to something Barro writes

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Production Function-Cobb-Douglas

- The Cobb-Douglas production function is:

$$Y_t = A_t K_t^{\alpha} L_t^{\beta}$$

- Taking logs,

$$\begin{aligned} \log(Y_t) &= \log(A_t K_t^{\alpha} L_t^{\beta}) \\ &= \log(A_t) + \alpha \log(K_t) + \beta \log(L_t) \end{aligned}$$

- Taking differences,

$$\begin{array}{rcl} \log(Y_t) & = & \log(A_t) + \alpha \log(K_t) + \beta \log(L_t)... \\ -\log(Y_{t-1}) & & -\log(A_{t-1}) + \alpha \log(K_{t-1}) + \beta \log(L_{t-1}) \\ & = & \log\left(\frac{A_t}{A_{t-1}}\right) + \alpha \log\left(\frac{K_t}{K_{t-1}}\right) + \beta \log\left(\frac{L_t}{L_{t-1}}\right) \\ \Delta^* Y_t & \approx & \Delta^* A_t + \alpha \Delta^* K_t + \beta \Delta^* L_t \end{array}$$

- Where Δ^* is "percent change in..."

Cobb-Douglas vs. Barro

- Our:

$$\Delta^* Y_t \approx \Delta^* A_t + \alpha \Delta^* K_t + \beta \Delta^* L_t$$

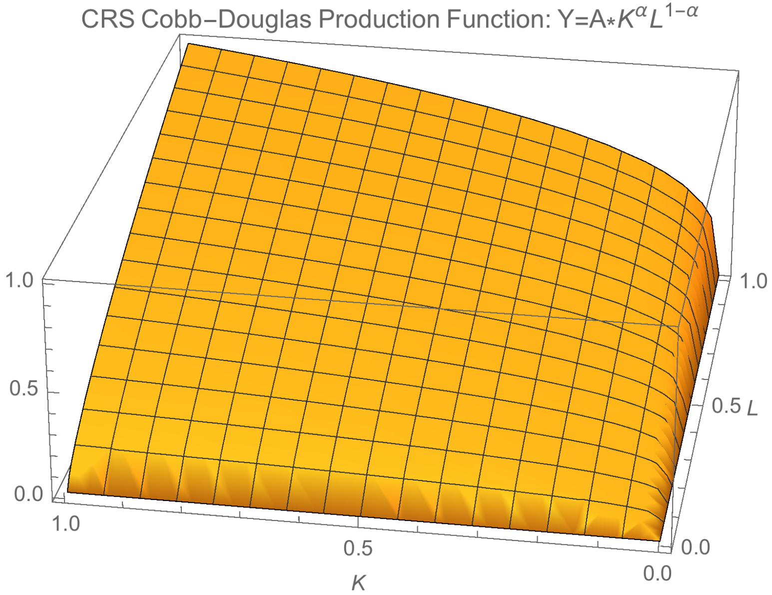
is Barro's

$$\frac{\Delta Y}{Y} = \frac{\Delta A}{A} + \alpha \frac{\Delta K}{K} + \beta \frac{\Delta L}{L}$$

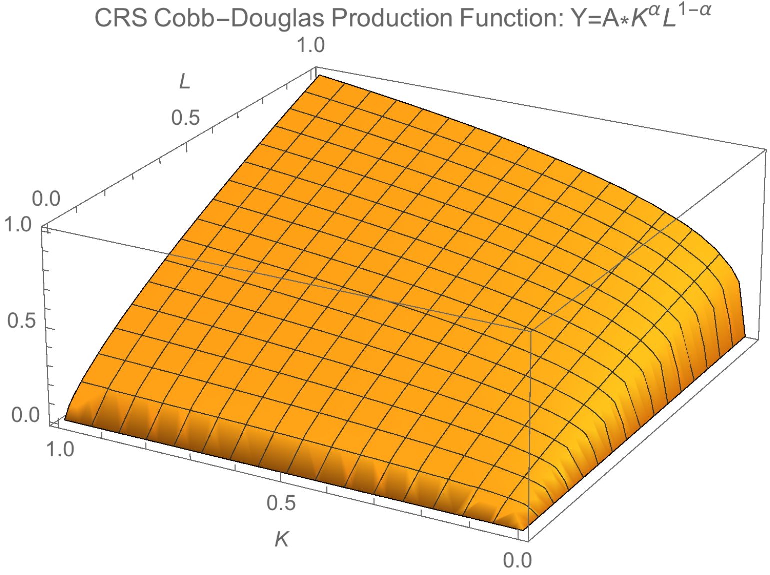
Cobb-Douglas vs. Barro

- If you're confused about what Barro says, use C-D. Everything will go through.
- Let's take a look at the Cobb-Douglas production function where $\alpha + \beta = 1$
- Three important pieces of information

Cobb-Douglas



Cobb-Douglas



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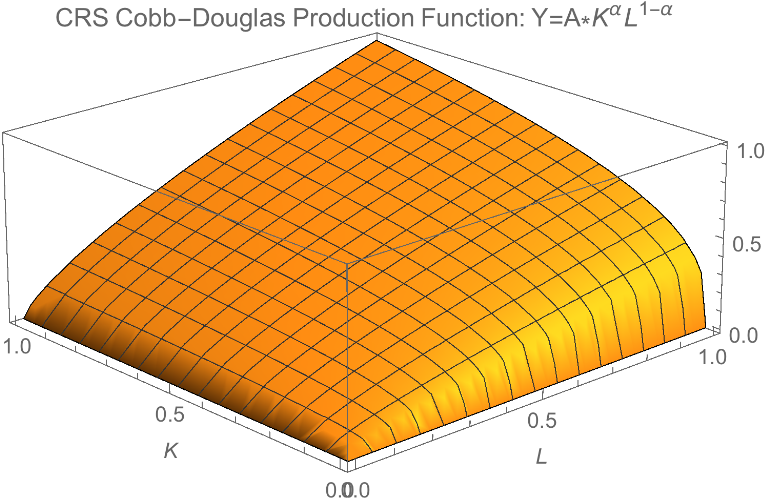

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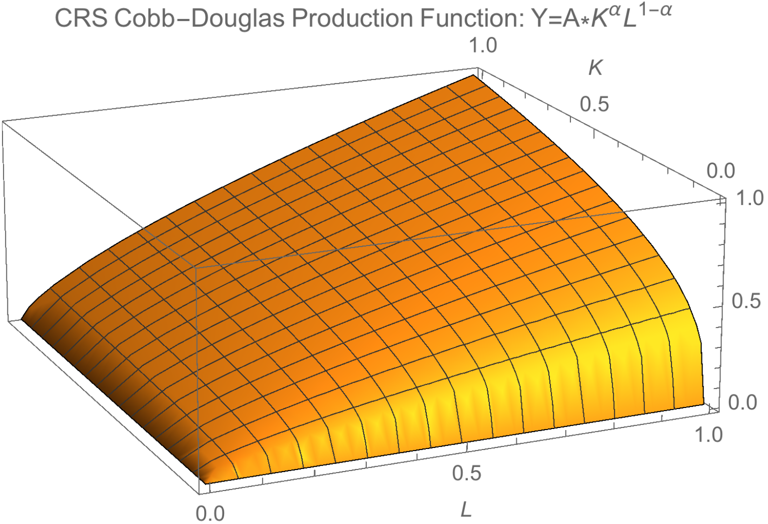

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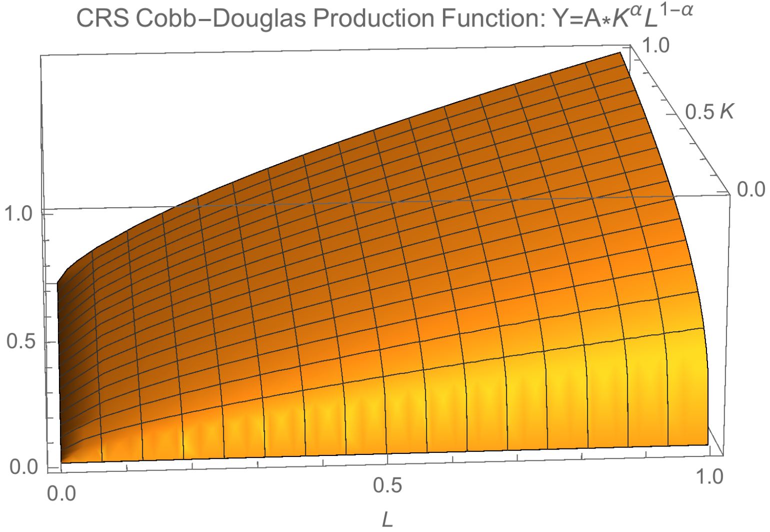

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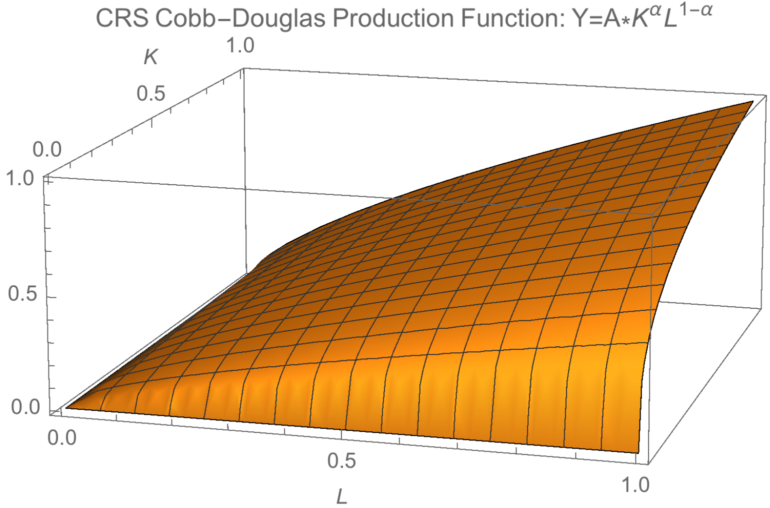

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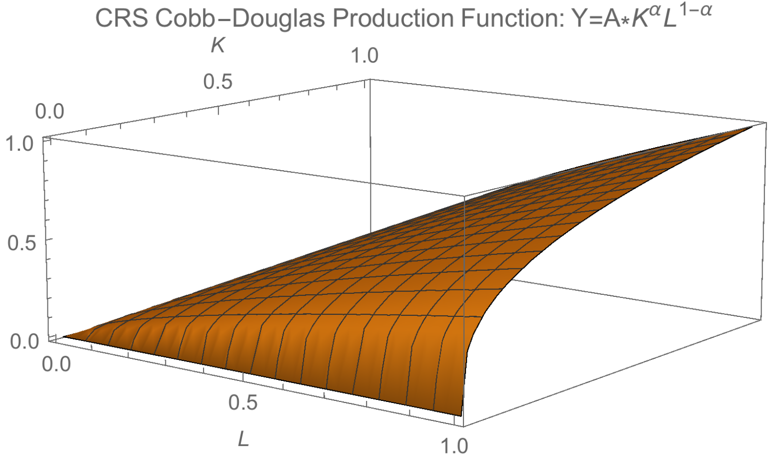

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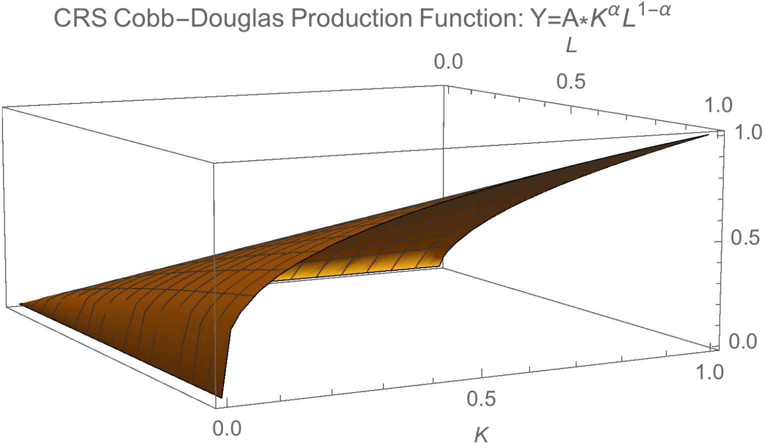

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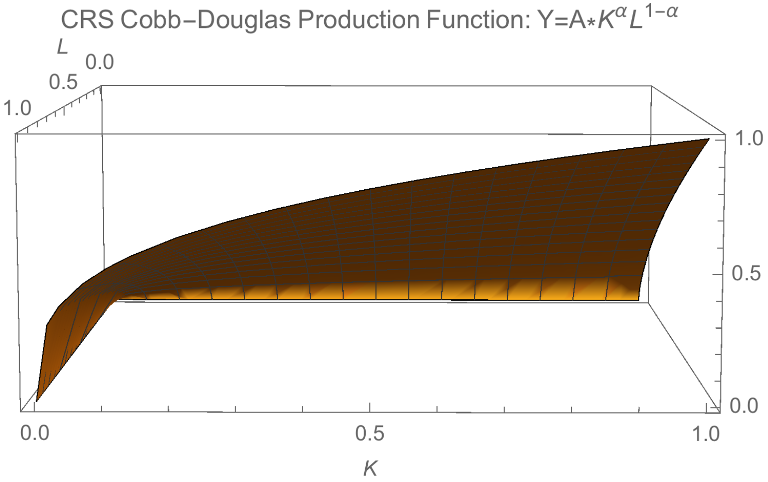

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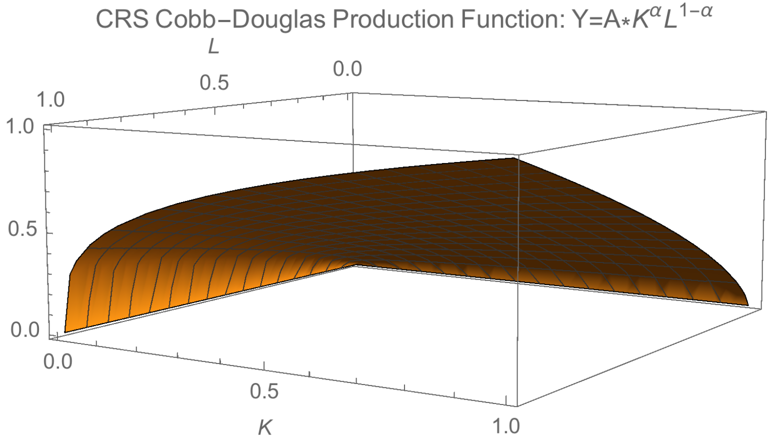

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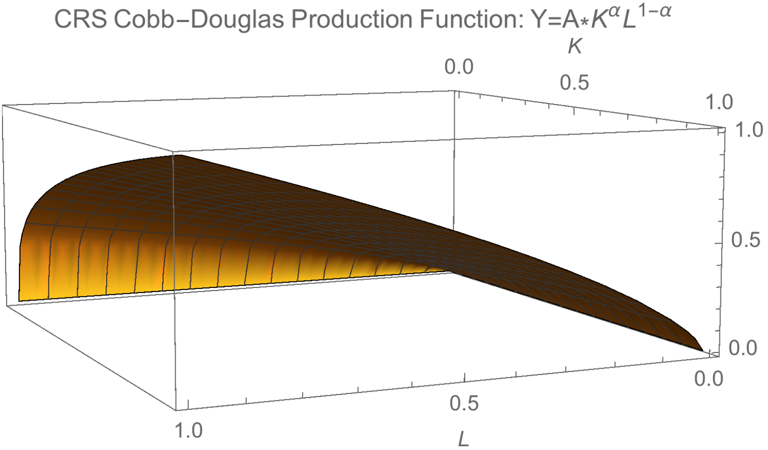

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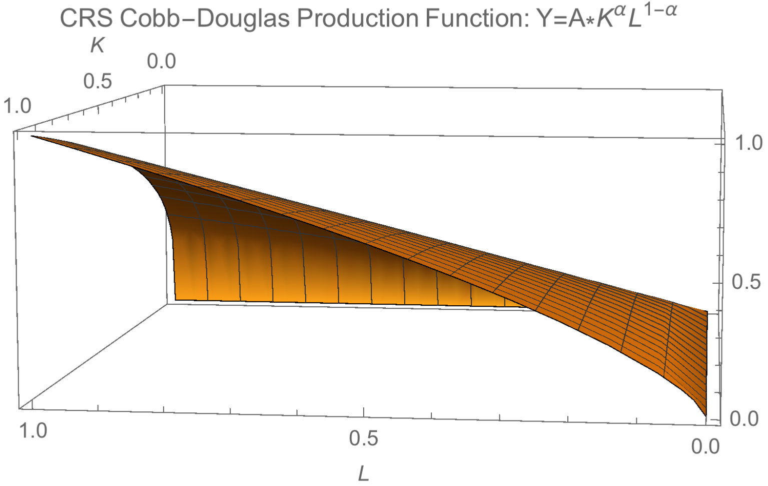

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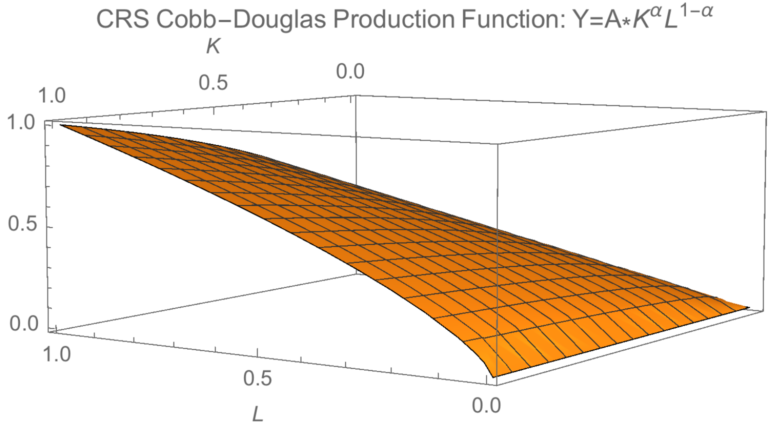

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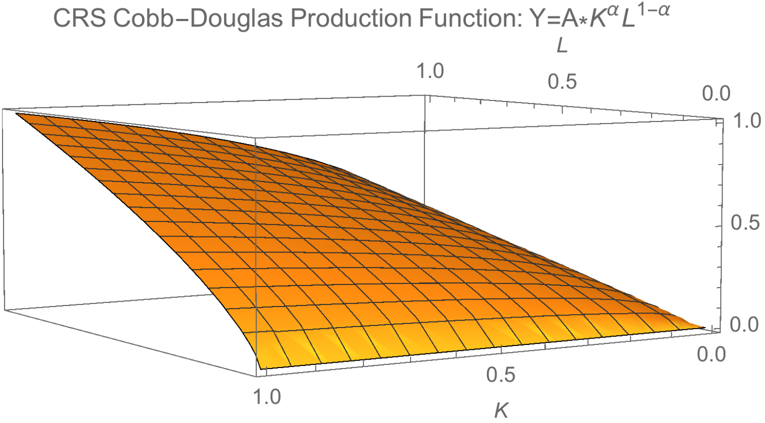

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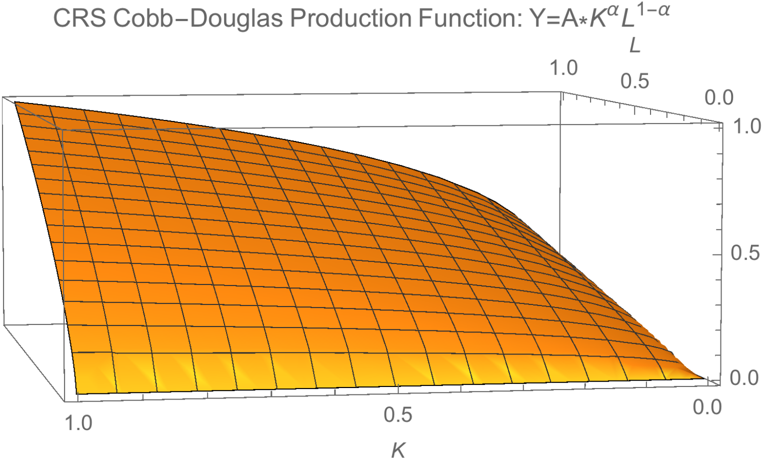

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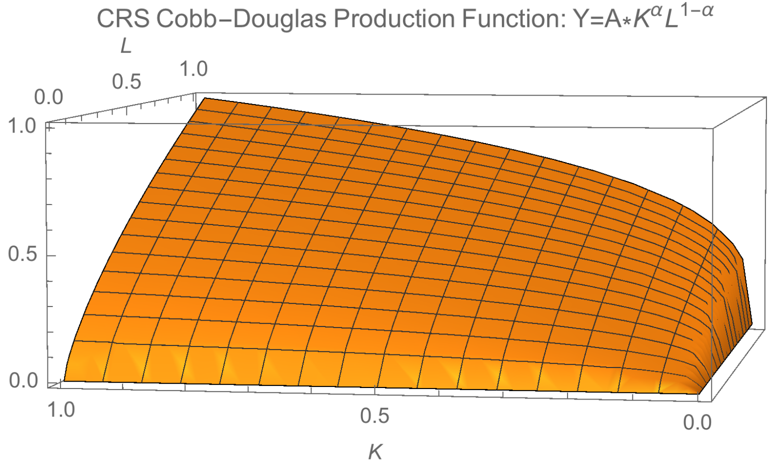

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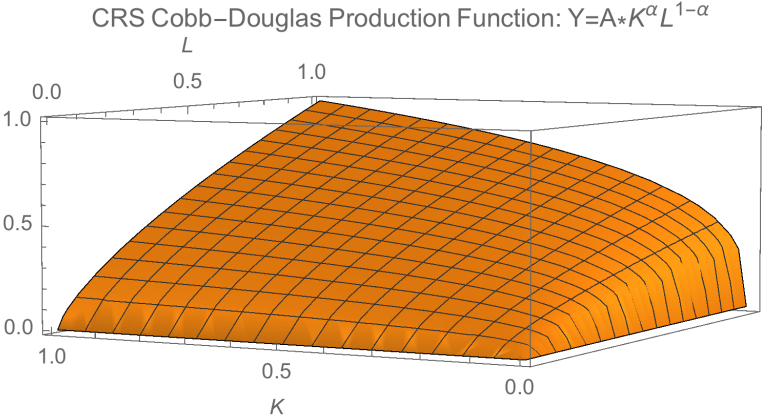
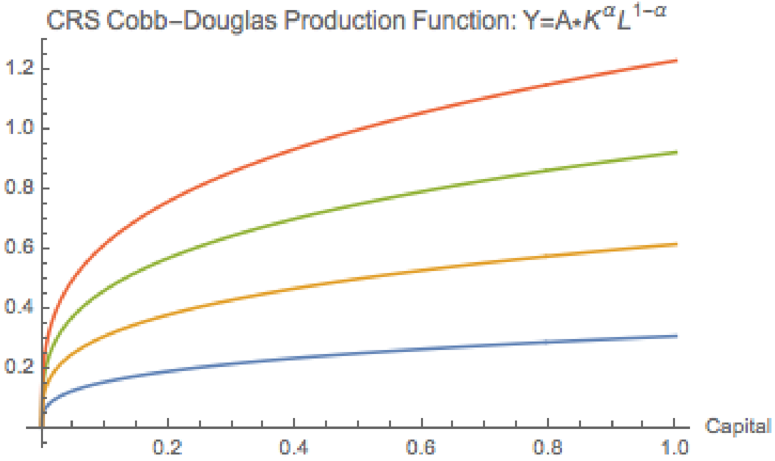
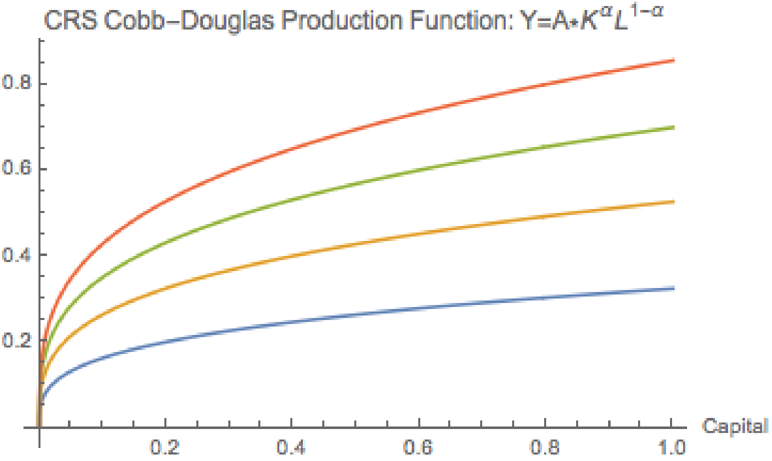


CD: L Constant, Raise A



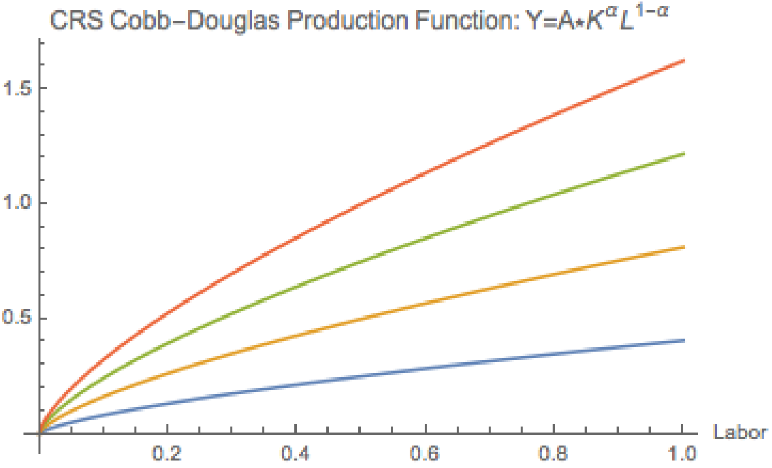
Note DRS as K rises, and A increases slope at a given K & increases level of Y at a given K.

CD: L Constant, Raise K



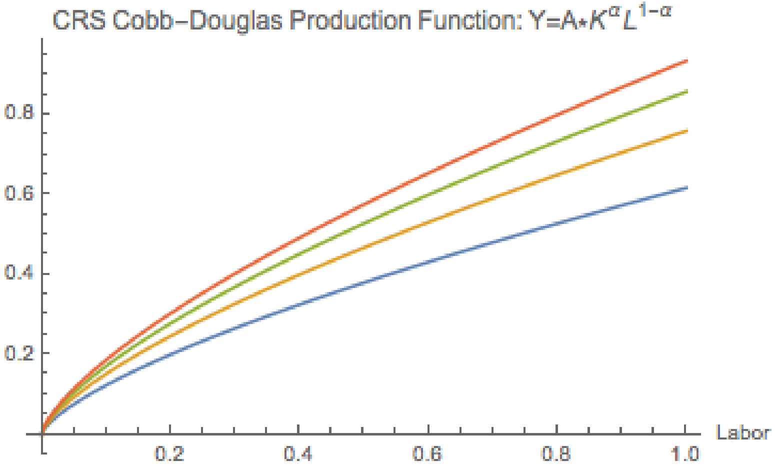
Note DRS as K rises, and L increases slope at a given K & increases level of Y at a given K.

CD: K Constant, Raise A



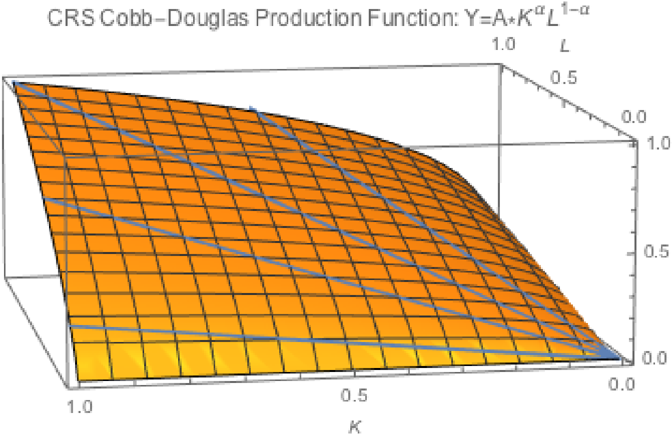
Note DRS as L rises, and A increases slope at a given L & increases level of Y at a given L.

CD: K Constant, Raise L



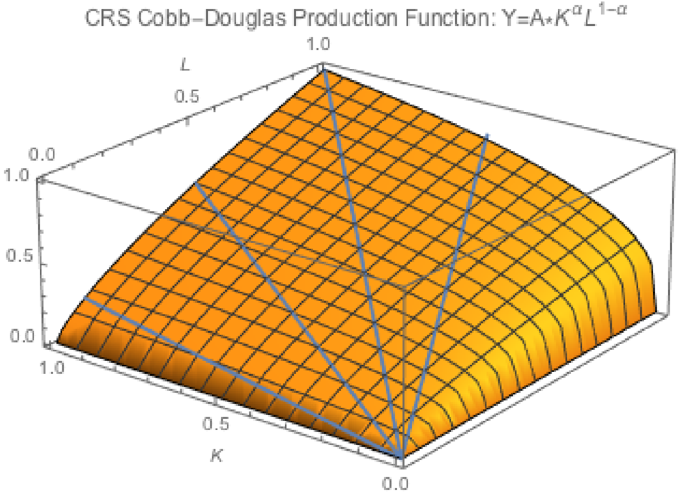
Note DRS as L rises, and K increases slope at a given L & increases level of Y at a given L.

CD: K/L Constant



Note that when K and L are increased in the same proportion, we are CRS.

CD: K/L Constant



Note that when K and L are increased in the same proportion, we are CRS.

Terminology

- For any given K, you could describe how a small change in L increases Y: this is the slope of Y with respect to L, holding K constant.
  - We can write this like: $\frac{Y(\bar{K},\bar{L})-Y(\bar{K},\bar{L}-\epsilon)}{\epsilon}$ for some small ϵ
  - Note that this is just $\left(\frac{\text{Rise}}{\text{Run}}\right)$
  - We call this slope the **Marginal product of labor** (MPL)
- For any given L, you could describe how a small change in K increases Y: this is the slope of Y with respect to K, holding L constant.
  - We can write this like: $\frac{Y(\bar{K},\bar{L})-Y(\bar{K}+\epsilon,\bar{L})}{\epsilon}$ for some small ϵ
  - We call this slope the **Marginal product of capital** (MPK)

Think about our pictures again

- Holding L and K constant, an increase in A increases the MPL and MPK
- Holding L constant, an increase in K increases the MPL and decreases the MPK
- Holding K constant, an increase in L increases the MPK and decreases the MPL
- A doubling of L and K doubles production and holds MPK and MPL the same as before!

Rewriting Cobb-Douglas

- Recall our functional form:

$$Y_t = A_t K_t^{\alpha} L_t^{1-\alpha}$$

- We can divide both sides by L to get:

$$\frac{Y_t}{L_t} = A_t \left(\frac{K_t}{L_t}\right)^{\alpha} \left(\frac{L_t}{L_t}\right)^{1-\alpha}$$

- Or, simplifying and writing per-unit labor as lowercase:

$$y_t = A_t k_t^{\alpha}$$

- This is really saying that it's all about the *per worker*, nothing special about larger populations

Growth Accounting

- Why are poor countries poor?
- Why is some Y high, some Y low?
- Cobb-Douglas points to big possibilities:
  - Poor countries are poor because they don't work (low L)
  - Poor countries are poor because they don't have capital (low K)
  - Poor countries are poor because they aren't productive (low A)
- Let's see

Growth Accounting-Roadmap: Next 10 Slides

1. Get an expression for percentage change in GDP in terms of percentage change in capital and percentage change in labor

2. Use that to get an expression for percentage change in GDP/worker as a function of percentage change in capital per worker

3. Combine savings behavior and capital's law of motion to get the percentage change in capital per worker as a function of current GDP, savings, and current capital.

4. Assume a growth rate of labor supply and find the growth rate of capital per worker

5. This gives us how GDP will grow per worker in the Solow Growth Model

Growth Accounting-I

- Recall that[1]

$$\Delta^* Y_t \approx \Delta^* A_t + \alpha \Delta^* K_t + \beta \Delta^* L_t$$

- If we are to declare Cobb-Douglas to be CRS, it must be that doubling both K_t and L_t doubles Y_t, holding A constant:

$$2 \approx 0 + 2\alpha + 2\beta$$

- Which gives:

$$\alpha + \beta = 1$$

- So CRS requires that the two exponents add to one
- Barro discusses how these mean that payments to factor inputs exhaust product.
- Let's understand why

[1] Where again, Δ^* are percentage changes!

Aside-I

- Let's imagine a firm has access to the production function:

$$Y_t(K_t, L_t) = A_t K_t^{\alpha} L_t^{1-\alpha}$$

- And has to pay labor wages wL_t and capital rental rates rK_t, so profit is:

$$\begin{aligned}\pi_t(K_t, L_t) &= Y_t(K_t, L_t) - w_t L_t - r_t K_t \\ &= A_t K_t^{\alpha} L_t^{1-\alpha} - w_t L_t - r_t K_t\end{aligned}$$

- Then first order conditions $\left(\frac{\partial \pi}{\partial K_t} = 0, \quad \frac{\partial \pi}{\partial L_t} = 0\right)$ will imply that:

$$(1-\alpha) = \frac{w_t L_t}{Y_t} \qquad \alpha = \frac{r_t K_t}{Y_t}$$

- Let's see why

Aside-II

$$(1-\alpha)A_t K_t^{\alpha} L_t^{-\alpha} - w_t = 0 \qquad \alpha A_t K_t^{\alpha-1} L_t^{1-\alpha} - r_t = 0$$

So:

$$(1-\alpha)\frac{L_t}{L_t} A_t K_t^{\alpha} L_t^{-\alpha} = w_t \qquad \alpha \frac{K_t}{K_t} A_t K_t^{\alpha-1} L_t^{1-\alpha} = r_t$$

Or:

$$(1-\alpha)\frac{Y_t}{L_t} = w_t \qquad \alpha \frac{Y_t}{K_t} = r_t$$

Rearranging:

$$(1-\alpha) = \frac{w_t L_t}{Y_t} \qquad \alpha = \frac{r_t K_t}{Y_t}$$

This is saying all production goes to someone! In this, we won't worry about entrepreneurs, they're either capital, labor, or both.

Aside-III

Recall that:

$$\log(x) - \log(\bar{x}) \approx \frac{x - \bar{x}}{\bar{x}}$$

For small deviations of x from $\bar{x}$.

Then:

$$\begin{aligned}
\Delta^* \left(\frac{y}{x}\right) &= \log\left(\frac{y}{x}\right) - \log\left(\frac{\bar{y}}{\bar{x}}\right) \\
&= \log(y) - \log(x) - \log(\bar{y}) + \log(\bar{x}) \\
&= \log\left(\frac{y}{\bar{y}}\right) - \log\left(\frac{x}{\bar{x}}\right) \\
&= \Delta^* y - \Delta^* x
\end{aligned}$$

That is, for small changes, the percentage change in a ratio is the percentage change in the numerator divided by the percentage change in the denominator.

Growth Accounting-II

- With our asides in mind, let's look at $\Delta^* y$, $y = \frac{Y}{L}$:

$$\Delta^* y \approx \Delta^* Y - \Delta^* L$$

- Similarly, for $k = \frac{K}{L}$,

$$\Delta^* k \approx \Delta^* K - \Delta^* L$$

- In other words, changes in GDP/worker comes either from changes in GDP, or changes in labor. Similarly, changes in capital per worker comes either from changes in capital, or changes in workers.
- Let's apply this

Growth Accounting-III

- We have, assuming no changes in productivity, that:

$$\Delta^* Y_t \approx \alpha \Delta^* K_t + (1-\alpha)\Delta^* L_t$$

- or:

$$\Delta^* Y_t \approx \alpha \Delta^* K_t + \Delta^* L_t - \alpha \Delta^* L_t$$

- Which becomes

$$\Delta^* Y_t - \Delta^* L_t \approx \alpha \left(\Delta^* K_t - \Delta^* L_t\right)$$

- Which becomes:

$$\Delta^* y_t \approx \alpha \Delta^* k_t$$

- Assuming no productivity growth, the percentage change in per-worker GDP is equal to the percentage change in per-worker capital!

Growth Accounting-IV

- Now we have an equation that relates GDP/worker as a function of capital/worker
- if the two line up, then we're done: we could state that rich countries are rich because they have a lot of productive capital
- So let's look at how we measure the capital stock

Growth Accounting-V

- We want to know what's driving changes in the stock of capital
- Assume that capital falls apart at the same rate δ
- That is, if $\delta = 0.01$, 1% of all capital fell apart each period
- An economy *makes* Y, and δK falls apart: net income is $Y - \delta K$
- So total real savings is: $s \cdot (Y - \delta K)$
- Where "s" is the real savings rate, the proportion of net income we save.

Growth Accounting-VI

- Net income $Y - \delta K$ either goes into consumption or real savings:

$$Y - \delta K = C + s(Y - \delta K)$$

- In a closed economy/no government, everything produced is either consumed or invested:

$$Y = C + I$$

- Combining the two, we get that:

$$\underbrace{I - \delta K}_{\text{Net investment}} = \underbrace{s(Y - \delta K)}_{Realsavings}$$

- This is a **crucial** insight!
- Demand=Supply
- Markets must clear
- There is no savings without investment

Growth Accounting-VII

- We have the relationship:

$$I - \delta K = s(Y - \delta K)$$

- We define the change in capital as:

$$\Delta K = I - \delta K$$

- We can put the two together:

$$\Delta K = s(Y - \delta K)$$

- Or, dividing by K, we get:

$$\Delta^* K = s\frac{Y}{K} - s\delta$$

- Now we have the growth rate of capital in terms of savings, GDP, and current capital.

- Let's turn to $\Delta^* L$

Growth Accounting-VIII

- We assume constant population growth in workers, so that:

$$\Delta^* L = n$$

- Recall that we can write the percentage change in per worker capital $\Delta^* k$ as:

$$\Delta^* k = \Delta^* K - \Delta^* L$$

- Plugging in our results and noting $Y/K = y/k$:

$$\Delta^* k = s\frac{y}{k} - s\delta - n$$

- This is our core result, so:

$$\Delta^* y_t \approx \alpha \Delta^* k_t$$

- Becomes:

$$\Delta^* y_t \approx \alpha \left(s\frac{y}{k} - s\delta - n \right)$$

Growth Accounting-IX

- We have our key Solow equation:

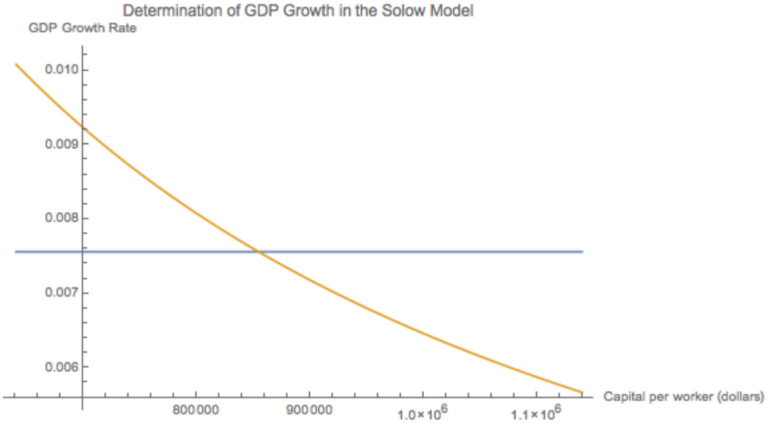
$$\Delta^* y_t \approx \alpha \left(s\frac{y}{k} - s\delta - n \right)$$

- Note that y/k is the *average* product of capital
- The average product of capital follows the marginal product of capital...
- So the average product of capital is diminishing as capital grows
- Let's graph out the two parts of the change in GDP/worker:

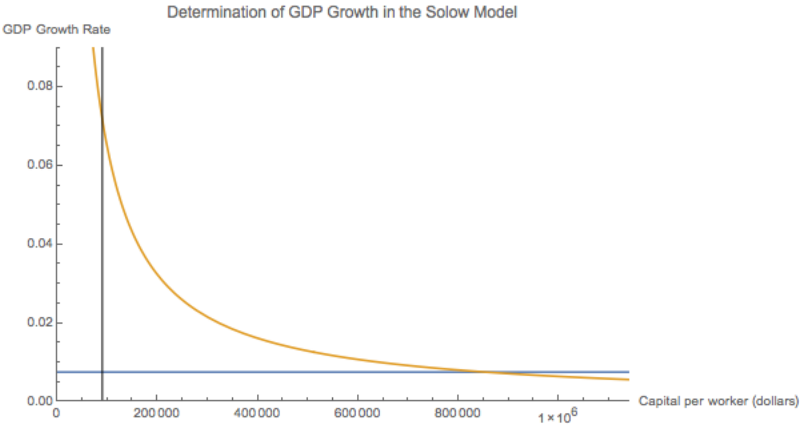
$$\alpha s\frac{y}{k} \quad vs. \quad \alpha(s\delta + n)$$

- For fun, $s = 0.19$, $\delta = 0.08$, $n = 0.01$, $\alpha = 0.30$, $y = 17$ trillion

Growth Accounting-X



Growth Accounting-XI



Growth Accounting-XII

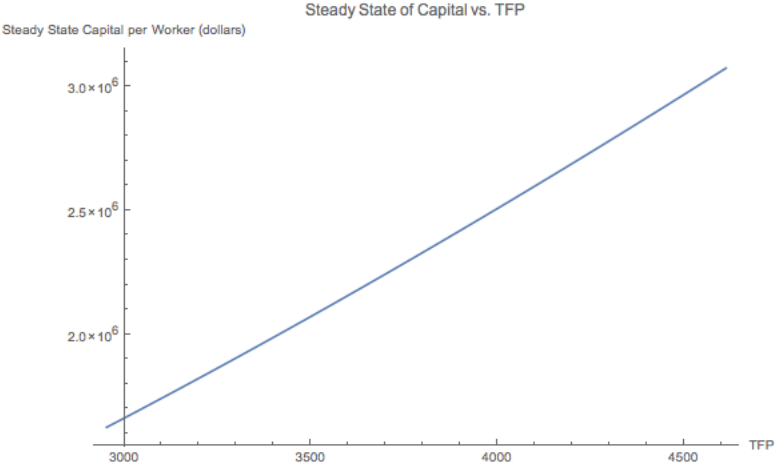
- Important points here
- First, understand the two influences:
  1. Depreciation of capital and population growth (decreasing the growth rate of GDP/capita) have a constant effect wrt how much capital there is
  2. Capital accumulation has a positive but declining effect increasing the growth rate of GDP/capita
  3. Where these two balance, we are at steady state, at k^* and therefore at y^*

Capital's Steady State-I

- There is a **steady state**, a level of capital after which all our savings are eaten up by depreciation
- We call the "steady" level of capital k^*.
- We can find it by setting growth equal to zero:

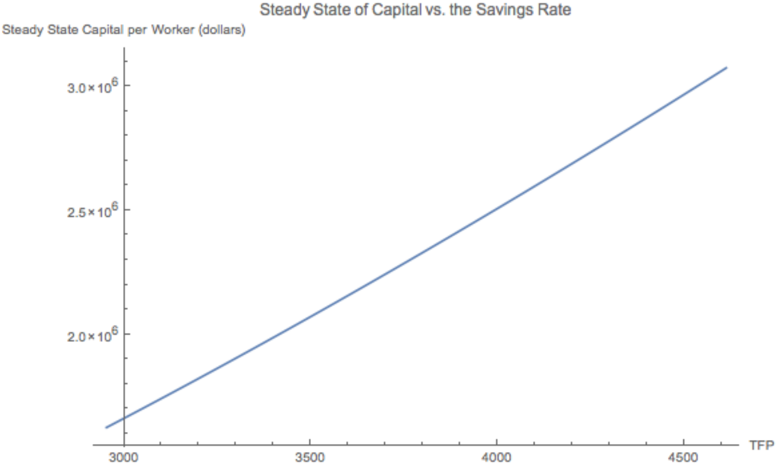
$$\begin{aligned} \Delta^* k &= s\frac{y^*}{k^*} - s\delta - n \\ 0 &= s\frac{Ak^{*\alpha}}{k^*} - s\delta - n \\ k^* &= \left(\frac{s\delta + n}{sA}\right)^{\frac{1}{\alpha - 1}} \end{aligned}$$

Capital's Steady State-II



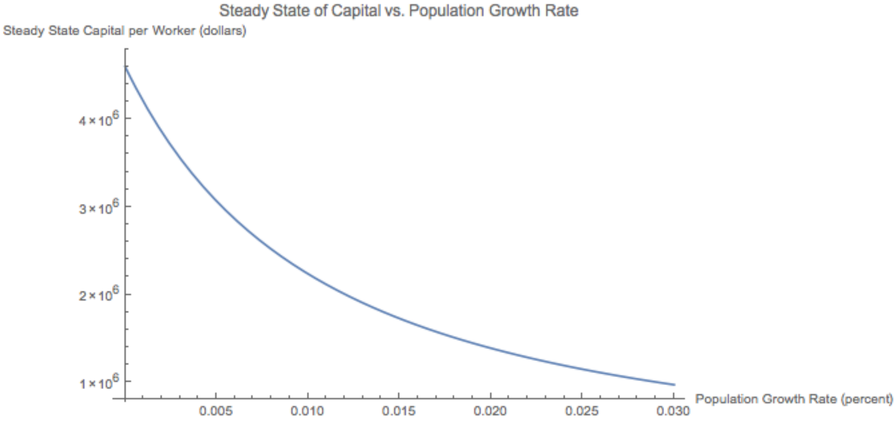
$$k^* = \left(\frac{s\delta + n}{sA}\right)^{\frac{1}{\alpha-1}}$$

Capital's Steady State-III



$$k^* = \left(\frac{s\delta + n}{sA}\right)^{\frac{1}{\alpha-1}}$$

Capital's Steady State-IIII



$$k^* = \left(\frac{s\delta + n}{sA} \right)^{\frac{1}{\alpha - 1}}$$

GDP/worker Steady State-I

- Note that GDP/capita closely mirrors capital per capita, with a slight change

Transitions to the Steady State-I

- We know what the steady state is
- How do we get there?
- Our whole economy is populated with unsurprised robots, so it's easy to calculate the path!
- Start with K_0, L_0, s, δ, and n
- Repeatedly plug these in to get evolution of paths

Transitions to the Steady State-II

- For Cobb-Douglas, we have:

$$\frac{y}{k} = \frac{Ak^{\alpha}}{k} \\ = Ak^{\alpha-1}$$

- So our formula[2]:

$$\Delta^* k = s\frac{y}{k} - s\delta - n$$

- Becomes:

$$\Delta^* k = sAk^{\alpha-1} - s\delta - n$$

- The percentage change in k is decreasing in k

[2]Note I divide both Y and K by L

Transitions to the Steady State-III

- How long does it take to get half the distance to the steady state?
- We can show that the rate of convergence is controlled by δ and n

Transitions to the Steady State-IV

- Recall:

$$\Delta^* k = s\frac{y}{k} - s\delta - n$$

- Plugging in $y = Ak^\alpha$

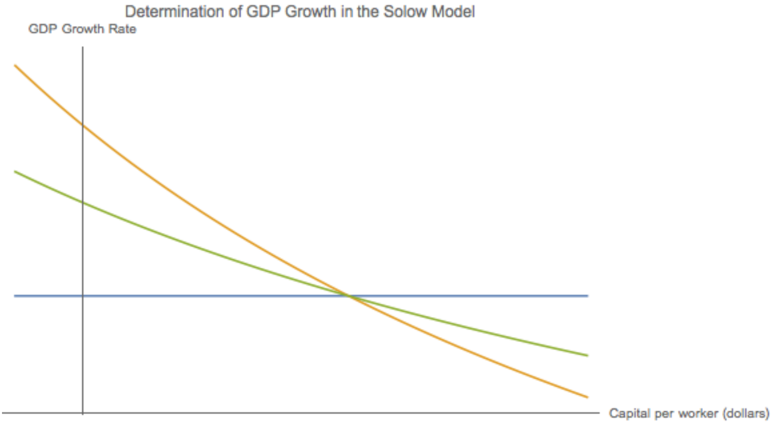
$$\Delta^* k = sAk^{\alpha-1} - s\delta - n$$

- Could alternatively write this as:

$$\frac{\partial \log k}{\partial t} = sA \exp(\log(k))^{\alpha-1} - s\delta - n$$

- This is a first-order differential equation that we can solve to get the time path of k wrt n, s, δ, A, and initial k.
- You don't have to know this

Capital's Steady State-IIII



What that causes convergence rates to differ here is α.